Math 630 Comprehensive Examination

8/22/2018

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Determine a compact SVD for the rank-one matrix $A = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix}$.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Define its condition number $\kappa_2(A)$ and show how it can be computed using the singular values of A.

   (c) Let $A \in \mathbb{C}^{m\times n}$. Prove that

$$\|A\|_2 = \max_{x\in\mathbb{C}^n,\, y\in\mathbb{C}^m,\, \|x\|_2=1,\, \|y\|_2=1} |y^* A x|.$$

   (Hint: Use the connection between $\|A\|_2$ and σ_1.)

2. Suppose $A \in \mathbb{C}^{m\times n}$ has the full column rank and $b \in \mathbb{C}^m$. Explain the difference between the full and and reduced QR factorization of A assuming that both factorizations can be written using

$$Q = \begin{bmatrix} \widehat{Q} & \widetilde{Q} \end{bmatrix} \qquad R = \begin{bmatrix} \widehat{R} \\ \widetilde{R} \end{bmatrix},$$

and state the basic properties of these matrices. Next. define

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} := Q^* b = \begin{bmatrix} \widehat{Q}^* b \\ \widetilde{Q}^* b \end{bmatrix}.$$

If $x \in \mathbb{C}^n$ is the solution of the least squares problem $\min_{x\in\mathbb{C}^n} \|b - Ax\|$ and $r = b - Ax$, prove

   (a) $Ax = \widehat{Q} b_1$ and $\widehat{R} x = b_1$,

   (b) $r = \widetilde{Q} b_2$ and $\|r\|_2 = \|b_2\|_2$.

3. (a) Define the Gauss-Seidel iteration process for solving a linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^n$. State (without proof) sufficient conditions on the matrix A under which the iteration converges to the solution regardless of b and the initial guess.

   (b) Consider the system

$$\begin{bmatrix} 4 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

   Decide whether the Gauss-Seidel process starting from the initial guess

$$x^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

converges to the solution of the system. If convergent, discuss the convergence rate (including the norm used). Compute the iterate $x^{(1)}$.

4. (a) Define the Cholesky factorization of a matrix $A \in \mathbb{R}^{n\times n}$, and state (without proof) a necessary and sufficient condition for a matrix to have a Cholesky factorization. What is the complexity (operation count) up to leading order of performing a Cholesky factorization of a generic (full) matrix?

   (b) Show that if a matrix A has a Cholesky factorization and is tridiagonal ($A_{ij} = 0$ if $|i-j| > 1$), then its Cholesky factor is also tridiagonal.

   (c) Show that if a matrix A has a Cholesky factorization with Cholesky factor R, then

$$\|R\|_2^2 = \|A\|_2.$$