PhD COMPREHENSIVE EXAM IN
PARTIAL DIFFERENTIAL EQUATIONS
January 2014

Do any 3 of the 4 problems. Show all work. Each problem is worth ten points.

Q1. Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set in the Hilbert space X with the inner product and norm written as $(\cdot,\cdot)$ and $\|\cdot\|$. The *Fourier coefficients* of $u \in X$ are defined as (u, x_n), $n = 1, 2, \ldots$.

(a) Prove Bessel's inequality: $\sum_{n=1}^{\infty}(u, x_n)^2 \le \|u\|^2$.
Hint: Consider $\|u - \sum_{n=1}^{N} a_n x_n\|^2$ where $a_n = (u, x_n)$.

(b) Consider the finite-dimensional subspace $Y = \text{span}\{x_1, \ldots, x_N\}$ of X. Given $u \in X$, define $v \in Y$ through $v = \sum_{n=1}^{N}(u, x_n)x_n$. Show that $\|u - v\| = \inf_{w\in Y}\|u - w\|$, that is, v is the closest point in the subspace Y to the point u.
Hint: Consider $\|u - \sum_{n=1}^{N} c_n x_n\|^2$ for arbitrary coefficients c_n.

Q2. Integrate the characteristic equations to obtain the classical solution (that is, before any shocks develop) to $x^3 u_x = u_y$ with $u(x, 0) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$. In what region in the $y > 0$ half-space is the solution defined?

Q3. A *spherical wave* is a solution of the form $u(r, t)$ of the 3D wave equation, where t is time and r is distance from the origin in $\mathbb{R}^3$. The 3D wave equation takes the form

$$u_{tt} = c^2\Big(u_{rr} + \frac{2}{r}u_r\Big) \tag{1}$$

in the case of spherical waves.

(a) Show that the general solution of (1) has the form

$$u(r, t) = \frac{1}{r}\big(F(r + ct) + G(r - ct)\big)$$

with arbitrary functions F and G.

(b) Obtain the (d'Alembert type) solution corresponding to the initial data $u(r, 0) = 0$ and $u_r(r, 0) = g(r)$, where g is an even function of r.

Q4. (a) What is the definition of the *weak derivative* $\frac{du}{dx}$ of a function $u \in C([0, 1])$? Prove that the weak derivative is unique a.e. on $(0, 1)$.

(b) Let $\Omega = (0, 1)$. What is the definition of the Sobolev space $H^1(\Omega)$?

(c) For $u \in H^1(\Omega)$, Ω as above, show that

$$\max_{x\in[0,1]} \big|u(x)\big|^2 \le u^2(0) + \|u\|_{H^1}.$$

Hint: Start with $2\int_0^x u u_x\, dx = \int_0^x (u^2)_x\, dx$.