MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2016

Do any three problems. Show all work. Each problem is worth ten points.

Q1 (a) Describe compactness of a set in a metric space in two different equivalent ways and in a third way in $\mathbb{R}^n$.

(b) Show that the union and intersection of two compact sets in a metric space are compact.

(c) Show that the algebraic sum of two compact sets in $\mathbb{R}^n$ is compact. (Algebraic sum of sets A and B in $\mathbb{R}^n$ is $A + B := \{a + b : a \in A, b \in B\}$.)

Q2 You are given the Riemann zeta function defined by the series

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^x},$$

for (real values of x) $x > 1$.

(a) Prove that for each $a > 1$, the series converges uniformly for $x \in [a, \infty)$.

(b) Discuss the continuity and differentiability of F for $x \in (1, \infty)$, providing rigorous justification.

(c) Prove that the series does not converge uniformly for $x \in (1, \infty)$.

Q3 Let $C([0, 1])$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ endowed with the supremum norm.

(a) Provide the definition of equicontinuity of a subset $K \subset C([0, 1])$.

(b) Prove that the closure of an equicontinuous set is equicontinuous.

(c) Prove that the sum of two equicontinuous subsets is equicontinuous. We note that the sum of two subsets A, B of $C([0, 1])$ is defined by

$$A + B = \{f + g \,|\, f \in A, g \in B\}.$$

(d) Prove by an example that the product of two equicontinuous subsets is not necessarily equicontinuous. Note that the product of two subsets A, B of $C([0, 1])$ is defined by

$$AB = \{fg \,|\, f \in A, g \in B\}.$$

Q4 Let n be a natural number with $n \geq 2$.

(a) Show that $\mathbb{R}^n \setminus \{0\}$ is path(=arcwise) connected.

(b) Let A be a path connected set in the matric space (M, d), and the function $f : (M, d) \to (N, \rho)$ be continuous on A. Show that $f(A)$ is path connected.

(c) Use (a)–(b) to show that the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \,|\, \|x\|_2 = 1\}$ is path connected.

(d) Let the function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ be continuous on $\mathbb{S}^{n-1}$. Suppose $g(x)$ is irrational for any $x \in \mathbb{S}^{n-1}$. Show that g is a constant function on $\mathbb{S}^{n-1}$.

Q5 Define Fréchet differentiability of a function from $\mathbb{R}^n$ to $\mathbb{R}$. Show that the following statements are equivalent:

(a) the functions $f(x)$ and $g(y)$ are differentiable on $\mathbb{R}$;

(b) the function $F(x, y) = f(x) + g(y)$ is Fréchet differentiable on $\mathbb{R}^2$;

(c) the function $G(x, y) = f(x + y) + g(x - y)$ is Fréchet differentiable on $\mathbb{R}^2$.

Note: You can use the fact that the composition of a differentiable function with a linear function is differentiable.