COMPREHENSIVE EXAMINATION

Math 650 – Optimization

January 1996

You must show all your work for full credit!

INSTRUCTIONS:

Do problems 1, 2, 5, and either 3 or 4, for a total of 4 problems.

Q1. Consider the linear program

$$\begin{array}{rl} \max & 4x_1 + 6x_2 + 3x_3 + x_4 \\ \text{subject to} & 3/2x_1 + 2x_2 + 4x_3 + 3x_4 \le 550 \\ & 4x_1 + x_2 + 2x_3 + x_4 \le 700 \\ & 2x_1 + 3x_2 + x_3 + 2x_4 \le 200 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0, x_4 \ge 0. \end{array}$$

(a) Solve the linear program above using the simplex method. (Bring into the basis the variable which makes the largest improvement per unit). What is the solution?

(b) Using the results of (a), determine the optimal solution to the dual of the above linear program.

(c) Now consider the linear program

$$\begin{array}{rl} \min & 2x_1 + 3x_2 \\ \text{subject to} & x_1 \ge 125 \\ & x_1 + x_2 \ge 350 \\ & 2x_1 + x_2 \le 600 \\ & x_1 \ge 0,\ x_2 \ge 0. \end{array}$$

Do *one* step of the big–M method towards finding a feasible solution of the above linear program. (Again, bring into the basis the variable which makes the largest improvement per unit).

(d) Determine (write explicitly) the dual linear program by using the Lagrangian method.

Q2. Consider the optimization problem

$$\begin{array}{rl} \max & x^2 + (y+1)^2 \\ \text{subject to} & -x^2 + y \geq 0 \\ & -x - y + 2 \geq 0 \end{array}$$

(a) Form the the Lagrangian and write the first order KKT conditions.

(a) Find the point(s) satisfying the first order KKT conditions.

(a) Sketch the feasible region, and graphically determine the optimal point(s).

(a) Is the second order KKT conditions satisfied at the optimal point(s)? Show all your steps in arriving at your answer.

Q3. Consider the optimization problem

$$\begin{aligned} \max \quad & \frac{1}{3}\sum_{i=1}^{n} x_i^3 \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 0 \\ & \sum_{i=1}^{n} x_i^2 = n \end{aligned}$$

(a) Write the Lagrangian for the problem as follows:

$$L(x, \lambda, \mu) = \frac{1}{3}\sum_{i=1}^{n} x_i^3 - \lambda \sum_{i=1}^{n} x_i + \frac{\mu}{2}(n - \sum_{i=1}^{n} x_i^2).$$

Find the KKT conditions.

(b) Show that $\lambda = 1$, and $\mu = (\sum_{i=1}^{n} x_i^3)/n$.

(c) Show that x_i can take only two values, x_- and x_+, where x_+ is positive and x_- negative. Find the possible values of x_+ and x_-.

(d) Find the optimal solution(s) to the problem, using (c).

(e) Investigate whether the second order KKT conditions are satisfied at the optimal solution(s) you found in (d).

Q4. Consider the well known Cauchy–Schwarz inequality

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}. \tag{1}$$

The purpose of this problem is to prove (1) using optimization techniques.

(a) Argue that (1) reduces to proving that the problem

$$\begin{aligned} \min \quad & \sqrt{\sum_{i=1} x_i^2} \cdot \sqrt{\sum_{i=1} y_i^2} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i y_i = 1 \end{aligned} \tag{2}$$

has optimal value 1. Then argue that in problem (2), we can consider the vector $y = \overline{y}$ fixed, so that it becomes equivalent the minimization problem

$$\begin{aligned} \min \quad & \sum_{i=1} x_i^2 \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i \overline{y}_i = 1. \end{aligned} \tag{3}$$

Show that, in order to prove (1), we need only prove (3) has optimal value

$$\frac{1}{\sum_{i=1}^{n} \overline{y}_i^2}. \tag{4}$$

(b) Now, solve (3) and justify all your steps. Show that the optimal value is the one given in (4).

(c) Show that the solution in (b) is unique, and use this result to characterize the equality case in (1).

(d) Show by direct calculation that the inequality (1) implies the triangular inequality

$$\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}.$$

5. Let A, B, and C be compact (bounded, closed) convex sets in $\mathbb{R}^n$. Suppose we have

$$A + C = B + C, \tag{5}$$

where, say $A+C$, is defined as $A+C = \{x_1 + x_2 : x_1 \in A, x_2 \in C\}$. You are going to show that (5) implies

$$A = B. \tag{6}$$

by going through the following steps:

(a) The *support function* s_A of A is defined as follows.

$$s_A(a) = \max\{\langle a, x\rangle : x \in A\} = \sup\{\langle a, x\rangle : x \in A\}.$$

Prove that the *epigraph* of s_A is a convex set.

(b) Show that (a) implies that s_A is a convex function.

(c) Show that

$$s_{A+C} = s_A + s_C. \tag{7}$$

Hint: Show separately that we have $s_{A+C} \leq s_A + s_C$ and $s_A + s_C \leq s_{A+C}$.

(d) Use (5) and (7) to show that $s_A = s_B$.

(e) Use a separation argument (and show and justify all your steps) to prove that

$$s_A \leq s_B \quad \Longrightarrow \quad A \subseteq B.$$

Consequently, conclude that (6) is true.