COMPREHENSIVE EXAMINATION
Math 650 / Optimization / January 2005
(Prepared by Dr. O. Güler)

Name ______________________________

INSTRUCTIONS:
You *must* solve Problem 2 (35 points) and Problem 4 (40 points). You must also solve *one* problem from the set $\{1, 3\}$ (25 points), but please *mark* clearly which of these two problems you would like to be graded.

1.(a) Show that for all values of a, the function $f(x, y) = x^3 - 3axy + y^3$ has no global minimizers or global maximizers.
(b) For each value of a, find all the critical point(s) of f and determine their status, that is, determine whether each critical point is a local minimizer, local maximizer, or a saddle point.

2. Consider the problem

$$\begin{aligned} \min \quad & -xy \\ \text{s.t.} \quad & x + y = 8, \\ & x \geq 0, y \geq 0. \end{aligned}$$

This is the problem of finding the rectangle of maximum area which has perimeter 16.

(a) write down the FJ (Fritz John) conditions, and show that $\lambda_0 \neq 0$, that is, KKT (Karush–Kuhn–Tucker) conditions are satisfied at all points satisfying the FJ conditions.

(b) show that the point $(x, y) = (4, 4)$ satisfies the KKT conditions.

(c) show that the point $(x, y) = (4, 4)$ satisfies the second order sufficient conditions. (Thus, the point $(x, y) = (4, 4)$ is a local optimizer to the problem.)

3. Consider the problem

$$\begin{array}{ll} \min & x^2 + (y-1)^2 \\ s.t. & -y + \frac{x^2}{K} \geq 0. \end{array}$$

For which values of the parameter $K > 0$ is the point $(x^*, y^*) = (0, 0)$ a local minimum? *Use second order conditions. It should be useful to sketch the problem.*

4. This problem is about duality concerning the projection of a point $a \in \mathbb{R}^n$ onto a linear subspace L in R^n. It has two parts.

(a) One way to formulate the above problem is to write $L = \{x : Ax = 0\}$ for an appropriate matrix and to consider the problem

$$\min \left\{ \frac{||x-a||^2}{2} : Ax = 0 \right\} \qquad (P_1).$$

(i) Formulate the Lagrange dual (D_1) of (P_1). State the Strong Duality Theorem as it applies to the pair (P_1) and (D_1). Prove that either this duality theorem holds true, or argue that it does not hold true.

(ii) Show that the dual problem (D_1) has a geometric interpretation as an orthogonal projection onto the range of A.

(b) A second way to formulate the same problem is

$$\min\{||x-a|| : x \in L\} \qquad (P_2)$$

(Here do *not* represent L in the form $Ax = 0$; just think of it as an implicit constraint.)

(i) Using the fact that $||u|| = \max_{||y|| \leq 1} \langle u, y \rangle$ for any $u \in \mathbb{R}^n$, write (P_2) as a minimax problem.

(ii) Write the dual (D_2) of the minimax problem above, and write it as

$$\max\{\langle a, y \rangle : ||y|| \leq 1, y \in M,$$

where M is a certain subset of $\mathbb{R}^n$. Give a description of M.