COMPREHENSIVE EXAM IN Math 650-Optimization
January 2017

You must solve **Q1, Q2, Q3** *and* **either Q4 or Q5**. *Mark clearly which of* **Q4** *and* **Q5** *you want to be graded. Each problem is worth ten points.*

Q1 Let $c \in \mathbb{R}^n$ be a unit vector, i.e., $\|c\|_2 = 1$. Consider the optimization problem on $\mathbb{R}^n$:

$$\min_{x \in \mathbb{R}^n} \; -x^T x \qquad \text{subject to} \qquad x^T x = c^T x + 2.$$

(a) Show that the constraint set is compact and the optimization problem has a global minimizer.

(b) Show that any Fritz-John point is a KKT point.

(c) Find all the KKT points, and determine minimizer(s) using the second-order optimality condition.

Q2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable convex function, and $C \subseteq \mathbb{R}^n$ be a closed convex set. Consider the convex minimization problem $\min_{x \in C} f(x)$.

(a) State the necessary and sufficient optimality condition in term of the variational inequality; no proof is needed.

(b) Suppose C is the affine set given by $\{Ev + d \,|\, v \in \mathbb{R}^m\}$ for a matrix $E \in \mathbb{R}^{n \times m}$ and a vector $d \in \mathbb{R}^n$. Use (a) to show that $x^* \in C$ minimizes f if and only if $E^T \nabla f(x^*) = 0$.

(c) Suppose C is a closed convex cone. Use (a) to show that $x^* \in C$ minimizes f if and only if $C \ni x^* \perp \nabla f(x^*) \in C^*$, where C^* is the dual cone of C.

Q3 Solve the following problems.

(a) Let $\{f_\alpha\}_\alpha$ be a (possibly infinite) family of convex functions on $\mathbb{R}^n$, i.e., each $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the pointwise supremum of f_α's, i.e., $g(x) := \sup_\alpha f_\alpha(x)$ for each $x \in \mathbb{R}^n$. Show that g is a convex function.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a (possibly non-differentiable) convex function, and $C \subseteq \mathbb{R}^n$ be a convex set. Show that a local minimizer $x^* \in C$ of f on C is a global minimizer of f on C.

(c) Consider the optimization problem on $\mathbb{R}^2$:

$$\min_{(x,y) \in \mathbb{R}^2} \; f(x,y) := (x+1)^2 + y^2 \qquad \text{subject to} \qquad 2x^2 - y \leq 0.$$

(i) Show that this optimization problem has a global minimizer.

(ii) Show that the function $f(x, y)$ is strictly convex, and the constraint set is convex. What can you say about the uniqueness of global minimizer? Justify your answer.

Q4 Solve the following problems.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping, i.e., $f(x) = Bx + d$ for some $B \in \mathbb{R}^{m \times n}$ and $d \in \mathbb{R}^m$, and let P be a convex polyhedron in $\mathbb{R}^m$. Show that $f^{-1}(P)$ is a polyhedron.

(b) Let K be a closed convex cone in $\mathbb{R}^n$, and $\Pi_K(x)$ denote the Euclidean projection of $x \in \mathbb{R}^n$ onto K. Show that $\Pi_K(x) = 0$ if and only if $-x$ belongs to the dual cone of K. (*Hint*: consider the VI associated with $\Pi_K(\cdot)$.)

Q5 Solve the following problems.

(a) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:

$$\mathbf{I}: \ Ax \geq b; \qquad \text{and} \qquad \mathbf{II}: \ y^T A = 0, \ y \geq 0, \ y^T b > 0.$$

(b) The support function of a nonempty set $S \subseteq \mathbb{R}^n$ is $\sigma_S(x) := \sup\{\langle x, z \rangle \mid z \in S\}$. Let F and G be two compact convex sets in $\mathbb{R}^n$ such that $\sigma_F(x) = \sigma_G(x), \forall\, x \in \mathbb{R}^n$. Show that $F = G$.