MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2017

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 Let (M,d) be a metric space and $\mathbb{R}^n$ denote the Euclidean n-space with the usual norm/metric. We shall use the metric ρ on $M \times \mathbb{R}^n$ given by

$$\rho((x_1,x_2),(y_1,y_2)) = d(x_1,y_1) + \|x_2 - y_2\|$$

where $\|.\|$ is the norm on $\mathbb{R}^n$.

The graph of a function $f : M \to \mathbb{R}^n$ is the subset of the product space $M \times \mathbb{R}^n$ given by

$$G(f) := \{(x, f(x)) : x \in M\}.$$

(a) If f is continuous, show that $G(f)$ is closed in $M \times \mathbb{R}^n$.

(b) If $f(M)$ is bounded in $\mathbb{R}^n$ and $G(f)$ is closed in $M \times \mathbb{R}^n$, show that f must be continuous.

Q2 Solve the following problems.

(a) A set C in a normed vector space V is called convex if for any $x, y \in C$, $\lambda x + (1-\lambda)y \in C$ for all real numbers $\lambda \in [0,1]$. Show that the closure of a convex set is convex.

(b) Let S be a connected set in a metric space (M,d). Suppose S contains more than one point. Show that every point of S is a limit point (also known as an accumulation point) of S.

(c) Let A be a bounded set in $\mathbb{R}^n$ with exactly two limit points x, y. **Use the open cover definition** to show that $A \cup \{x,y\}$ is a compact set.

Q3 Consider the series

$$\sum_{n=1}^{\infty} \frac{x^2}{n^3} \sin\left(\frac{n^2}{x^2}\right),$$

where $x \in (0,\infty)$.

(a) Prove that the series converges uniformly on $(0,a]$ for each $a > 0$.

(b) Explain why the sum is well defined and continuous on $(0,\infty)$.

(c) Prove that the series does not converge uniformly on $(0,\infty)$.

Q4 (a) Provide the definition of equicontinuity for a set S consisting of functions $f : [0,1] \to \mathbb{R}$.

(b) Let $C([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the supremum norm.

Define the map $J : C([0,1]) \to C([0,1])$ by

$$J(f)(x) = \int_0^x f(t)dt, \quad x \in [0,1],$$

for all $f \in C([0,1])$. Prove that if $S \subset C([0,1])$ is bounded (in the sup-norm metric) then its direct image $J(S) \subset C([0,1])$ is both a bounded and equicontinuous subset of $C([0,1])$.

(c) With J as defined above, provide an example of a set $S \subset C([0,1])$ such that S is equicontinuous, but $J(S)$ is not equicontinuous.

Q5 (a) Provide the definition of the Frechet derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are normed vector spaces (possibly infinite dimensional).

(b) Let $C([0,1])$ be the space of continuous real valued functions on $[0,1]$ endowed with the supremum norm. Let $k \in C([0,1])$ be a fixed continuous function. Define $F : C([0,1]) \to \mathbb{R}$ by

$$F(f) = \frac{1}{2}\int_0^1 (f(x))^2 dx - \int_0^1 k(x)f(x)dx,$$

for all $f \in C([0,1])$. Show directly from the definition that F is Frechet differentiable on the entire domain and compute the Frechet derivative $DF(f)$.

(c) For the F defined above, find all possible choices of $f \in C([0,1])$ such that $DF(f) = 0$.