MASTER'S COMPREHENSIVE EXAM IN
Math 603 -MATRIX ANALYSIS
January 2017

Q1 Solve the following problems.

(a) Let S be a subspace of the vector space V, and a vector $z \in V$. Show that $z+S$ is a subspace if and only if $z \in S$. What is the relation between $z+S$ and S when $z \in S$?

(b) Let A be a 8×8 *diagonalizable* complex matrix with three distinct eigenvalues λ_1, λ_2, and λ_3. Let E_i denote the eigenspace associated with λ_i for $i = 1, 2, 3$. Suppose $\dim E_1 = 1$ and $\dim E_2 = 3$. Determine the algebraic multiplicity of λ_3, and find the determinant of A^T in terms of λ_1, λ_2, and λ_3. Justify your answers.

Q2 For $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathcal{S} \subseteq \mathbb{R}^{n\times 1}$, the set of n-dimensional column vectors, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{A}x | x \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. Prove that

(a) If $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times 1}$, $\mathbf{A}(\mathcal{S})$ is a subspace of $\mathbb{R}^{m\times 1}$.

(b) If $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times 1}$ and $\mathcal{S} \cap N(\mathbf{A}) = \{\mathbf{0}\}$, where $N(\mathbf{A}) = \{x | \mathbf{A}x = \mathbf{0}\}$, then $\dim \mathbf{A}(\mathcal{S}) = \dim(\mathcal{S})$.

Q3 Let $A \in R^{m\times n}$, $B \in R^{n\times m}$, with $m \geq n$. Show that AB and BA have the same set of nonzero eigenvalues, that is, show that $\sigma(AB) = \sigma(BA) \cup \{0, ..., 0\}$, where there are $m-n$ zeros.

Q4 Answer the following questions:

(a) Is $A = \begin{pmatrix} 5+i & -2i \\ 2 & 4+2i \end{pmatrix}$ as normal matrix? Justify.

(b) Show that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is normal and has real eigenvalues if and only if $\mathbf{A}$ is symmetric.

(c) Justify: A triangular matrix is normal if and only if it is diagonal.

Q5 Solve the following problems.

(a) N is *nilpotent of index* k when $N^k = 0$ but $N^{k-1} \neq 0$. If N is a nilpotent operator of index n on $\mathbb{R}^n$, and if $N^{n-1}(y) \neq 0$ for some $y \in \mathbb{R}^n$, show $\mathcal{B} = \{y, N(y), N^2(y), \ldots, N^{n-1}(y)\}$ is a basis for $\mathbb{R}^n$, and then demonstrate that the matrix $[N]_{\mathcal{B}}$ of N with respect to the basis $\mathcal{B}$ has the form

$$[N]_{\mathcal{B}} = J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

(b) If A and B are any two $n \times n$ nilpotent matrices of index n, explain why $A \simeq B$.

(c) Explain why all $n \times n$ nilpotent matrices of index n must have trace zero and rank $n-1$.