Math 620 Qualifying Exam 1/19/2012

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1a) Derive a numerical integration formula of the form

$$\int_{-h}^{h} f(x)dx \approx Af(0) + Bf'(-h) + Cf''(h).$$

(In other words, find A, B, and C.)

(b) For how high a degree polynomial is this formula exact? (Note: to get credit you must *show* that the highest degree polynomial integrated exactly is what you claim.)

(c) We want to determine the Gaussian quadrature formula with 3 nodes and 3 weights for the integral

$$\int_{-1}^{1} f(x)dx.$$

Using the theory of Gaussian quadrature, find the nodes for this integration rule.

(d) Assuming we are able to find the formula in Part (c) above, what is the highest degree polynomial for which this formula will be exact?

(2a) Give the Taylor series derivation of the scalar version of Newton's Method and from this derivation show that Newton's Method is quadratically convergent.

(b) What is the main condition required for Newton's Method to converge?

(c) Let α be the smallest positive root of

$$f(x) = 1 - x + \sin x.$$

Find an interval $[a, b]$ containing α for which the bisection method will converge.

(d) Estimate the number of iterations of bisection needed to find α within an accuracy of 5×10^{-8}.

(3) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and denote by f_n the solution to the standard polynomial least squares problem

$$\min_{p \in \mathcal{P}_n} \int_a^b |f(x) - p(x)|^2 dx \ , \tag{1}$$

where $\mathcal{P}_n$ is the space of polynomials of degree at most n.

(a) Show that the graphs of f_0 and f intersect at least once, and the graphs of f and f_1 intersect at least twice. (This shows that least-squares approximations are in fact interpolants with certain nodes.)
Hint: Use an argument similar to the one used for showing that orthogonal polynomials have real, distinct roots.

(b) Verify point (a) above for $f(x) = x^2$ on $[0, 1]$ (find f_0, f_1 and the intersection points).

(4) Consider the initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \ , \tag{2}$$

together with the following multistep numerical method for solving (2):

$$y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(x_{n+1}, y_{n+1}) \ , \tag{3}$$

where h is the step size, and $x_n = x_0 + hn$, $n = 0, 1, 2, \ldots$.

(a) Compute the order of *local truncation error* $\tau_n(Y) = T_n(Y)/h$ for in the method (3) (you may use either Taylor's formula or the more general result on multistep methods), and discuss its stability and convergence properties (including order of convergence, if so).

(b) For $f(x, y) = ay$ and $y_0 = 1$ (so the solution is $y(x) = \exp(ax)$), compute y_2, given $y_1 = \exp(ah)$.